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# Efficient Algorithm Based on the Woodbury Formula for Modeling Multi-port Antenna Systems

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This paper presents an efficient computational approach for calculating the characteristics of multi-port antenna systems using boundary integral equations. Modeling of antenna radiation is based on finding surface currents placed on the antenna structural elements. Numerical solution of integral equations for unknown currents is carried out by the Galerkin method using RWG basis functions. The antennas excitation is modeled by placing a system of lumped ports. A key challenge addressed is the calculating of mutual coupling characteristics (mutual impedance matrices, S-parameters) for various conditions of ports loading. This requires multiple solutions of a system of linear equations. In each case, the system matrix undergoes a change in blocks consisting of elements responsible for the interaction of the antenna ports with each other and with surface currents. To overcome this, an algorithm based on the Woodbury formula is developed, significantly reducing computational costs by leveraging the low-rank nature of port-related matrix modifications. The method's effectiveness is demonstrated for both wire and patch antenna arrays, showing substantial speedups – approximately proportional to the number of ports for direct solvers and significant gains for iterative solvers using mosaic-skeleton approximations while maintaining solution accuracy.

*Keywords: matrix methods, numerical algorithms, computational electrodynamics, antenna radiation, boundary integral equations, Woodbury formula.*

## Introduction

The Method of Moments (MoM) approach allows for the simulation of the performance of antennas with metal and dielectric parts. In case of metallic antennas, the electromagnetic field has an integral representation via surface currents on their metal parts, and the problem of antenna radiation can be treated as the scattering problem of some primary field generated by the antenna itself. One common mathematical model for such fields is the lumped port model, also known as the delta-gap model [2]. With a known primary field, MoM reduces antenna simulation to a system of surface integral equations [2, 13]. Some common performance characteristics, such as impedance, S-parameters, and the Voltage Standing Wave Ratio (VSWR), can be expressed in terms of electric currents, which are the solution of these integral equations.

MoM holds an advantage over volume-discretization techniques, as it does not require meshing the entire problem space, only the surfaces. Furthermore, it inherently satisfies the radiation condition at infinity.

In multi-port systems, mutual coupling between ports significantly affects the overall radiation characteristics. Matrices of mutual impedance and S-parameters are central to the analysis of this coupling. These matrices are computed through a series of numerical simulations, where individual ports are activated sequentially as active sources while the others are terminated with passive loads (e.g., matched feeder lines). This approach requires one simulation for each active port, leading to specific computational challenges.

A well-known challenge of MoM is that discretizing the integral equations yields linear systems with large dense matrices. Solving these systems is the dominant computational cost of the numerical simulation.

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For antenna systems, the main part of the system matrix describes the electromagnetic interaction between surface discretization elements (e.g., mesh cells). However, some rows and columns are modified according to the current active port. Thus, computing the complete mutual coupling matrices involves solving a series of linear systems, each with a modified system matrix. Such matrices can be represented as the sum of a fixed matrix (from the structure) and a low-rank correction per port configuration. A naive approach, solving each modified system independently, results in a computational cost proportional to the number of ports.

To address this inefficiency, we propose an algorithm based on the Woodbury matrix identity [3]. This identity allows for the efficient computation of the inverse of a matrix after it has been modified by a low-rank update. For a system of $n$ equations, a direct solution via LU decomposition requires $\Omega = O(n^3)$ operations and iterative solution requires $\Omega = O(n^2 I_0)$ operations ($I_0$ is the number of iterations). Solving for $m$ ports naively would therefore scale as $m\Omega$. Our algorithm leverages the Woodbury formula to obtain the exact solution for all port configurations by solving only one linear system with $m$ right-hand sides. This requires $\Omega + O(nm^2 + m^3)$ operations, making the time for the multi-port solution comparable to that of a single simulation when $m \ll n$.

For systems with a moderate number of unknowns (up to approximately 40,000), we employ a direct solver. For larger problems, we use an iterative solver – the Generalized Minimal Residual Method (GMRES) [7] – accelerated by performing matrix-vector multiplications in the mosaic-skeleton format [10]. This format is a data-sparse approximation that operates on a compressed representation of the dense matrix [11]. The proposed method applies both direct and iterative solvers. We present computational examples of antenna arrays with numerous feed points, demonstrating the significant efficiency gains achieved by our method.

The article is organized as follows. The first section contains the statement of the problem, the boundary integral equation method, the numerical scheme and the description of antenna model. The algorithm based on the Woodbury matrix identity is also described in this section. The second section includes the testing of the developed algorithm on two examples: wire antenna array and patch antenna array. The discussion of the results is given in the conclusion section.

## 1. Mathematical Model

We consider a perfectly conducting antenna system. The antenna system includes a set of radiating elements connected to a feeder. Each such element can operate either for radiation or for signal reception.

Let $\Sigma$ denote the union of all perfectly conducting surfaces forming the antenna structure. We consider the problem of antenna system radiation as a scattering problem. The primary field is created by a system of antenna ports. The antenna excitation is described in subsection 1.3.

Let us consider the general formulation of the problem of scattering a given primary field for monochromatic wave [13]. The electric and magnetic fields are sought in the form:

$$\vec{E}(x,t) = \mathbf{E}(x)e^{-\mathbf{i}\omega t},\ \vec{H}(x,t) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\mathbf{H}(x)e^{-\mathbf{i}\omega t}, \tag{1}$$

where $\varepsilon_0$ and $\mu_0$ are the electric and magnetic constants, $\omega$ is the angular frequency. The problem is reduced to the spatial components of the electric and magnetic fields $\mathbf{E}(x)$ and $\mathbf{H}(x)$.

The surrounding medium is assumed to be isotropic, homogeneous, and without conductivity. The electromagnetic field everywhere outside the antenna structural elements is described by Maxwell's equations. In the monochromatic case they have the following form:

$$\begin{aligned} \operatorname{rot}\mathbf{E} &= \mathbf{i}\omega k\mathbf{H}, \\ \operatorname{rot}\mathbf{H} &= -\mathbf{i}\omega k\mathbf{E}, \end{aligned} \tag{2}$$

here $k$ is the wave number, defined by the formula $k = \omega\sqrt{\varepsilon_0\mu_0}$.

The magnetic field can be excluded from equation (2). The equations for the electric field are:

$$\Delta\mathbf{E} + k^2\mathbf{E} = 0, \quad \operatorname{div}\mathbf{E} = 0, \tag{3}$$

provided that the magnetic field has the form

$$\mathbf{H} = \frac{\operatorname{rot}\mathbf{E}}{\mathbf{i}k}. \tag{4}$$

We assume that the total electric and magnetic fields have the form [2, 13]

$$\mathbf{E}_{tot} = \mathbf{E}_{inc} + \mathbf{E}, \quad \mathbf{H}_{tot} = \mathbf{H}_{inc} + \mathbf{H}. \tag{5}$$

Here $\mathbf{E}_{inc}(x)$ and $\mathbf{H}_{inc}(x)$ are the primary electric and magnetic fields created by the antenna excitation, $\mathbf{E}(x)$ and $\mathbf{H}(x)$ are the unknown secondary electric and magnetic fields. These secondary fields must satisfy equations (3)–(4) outside the antenna structural elements. The total field must satisfy the boundary condition on the antenna surface $\Sigma$:

$$n \times \mathbf{E}_{tot} = 0, \quad x \in \Sigma. \tag{6}$$

Also the secondary fields have to satisfy the Sommerfeld radiation condition at infinity:

$$\mathbf{E}(x) = O\left(\frac{1}{|x|}\right), \quad \frac{\partial\mathbf{E}(x)}{\partial|x|} - ik\mathbf{E}(x) = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty. \tag{7}$$

$$\mathbf{H}(x) = O\left(\frac{1}{|x|}\right), \quad \frac{\partial\mathbf{H}(x)}{\partial|x|} - ik\mathbf{H}(x) = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty. \tag{8}$$

### 1.1. Method of Moments

In this subsection we describe the integral representation for the electric field and the integral equation system [2].

The electric field:

$$\mathbf{E}(x) = \frac{\mathbf{i}}{k_0}\mathcal{K}[\Sigma, \boldsymbol{g}](x), \quad x \in \mathbb{R}^3 \setminus \Sigma, \tag{9}$$

where $\mathcal{K}[\Sigma, \boldsymbol{g}]$ is the electric field operator:

$$\mathcal{K}[S, \boldsymbol{g}](x) = \operatorname{rot}\operatorname{rot}\int\limits_S \boldsymbol{g}(y)F(x-y)d\sigma_y. \tag{10}$$

In formula (10), $S$ is random surface, $\boldsymbol{g}$ is a tangential vector field defined on the surface $S$, $F$ is the Green's function $F(x-y) = \frac{e^{\mathbf{i}k|x-y|}}{|x-y|}$.

In expression (9), $\boldsymbol{g}$ is the unknown surface current.

The integral operator $\mathcal{K}$ in (10) is defined everywhere outside the integration domain $S$. The field defined by this operator for a sufficiently smooth function $\boldsymbol{g}$ has boundary values on both sides of the surface $S$, for which the formula [15] is:

$$\mathcal{K}^{\pm}[S,k,\boldsymbol{g}](x) = \mathcal{K}[S,k,\boldsymbol{g}](x) \pm 2\pi\,\boldsymbol{n}(x)\,\mathrm{Div}\,\boldsymbol{g}(x), \quad x \in S, \tag{11}$$

where $x$ is a point of smoothness of the surface, not lying on its edge, $\boldsymbol{n}(x)$ is the unit vector of the positive normal at point $x$, the signs $\pm$ correspond to the boundary values from the side of the normal vector (from the positive side of the surface) and from the opposite side (from the negative side of the surface), respectively. Here $\mathcal{K}[S,k,\boldsymbol{g}](x)$ is the direct value of the operator on the surface. It arises if the value of the point $x \in S$ is substituted directly into the expression:

$$\mathcal{K}[S,\boldsymbol{g}](x) = \int\limits_S \mathrm{rot}\,_x \mathrm{rot}\,_x(\boldsymbol{g}(y)F(x-y))d\sigma_y. \tag{12}$$

In this case the hypersingular integral on the right-hand side is a Hadamard finite part integral.

We substitute expression (9) into the boundary condition (6) and apply the formula (11) for the boundary values of the integral operator (11) to obtain the boundary integral equation for the unknown function $g \in \Sigma$:

$$\frac{\mathbf{i}}{k_0}\boldsymbol{n}(x) \times \mathcal{K}[\Sigma,\boldsymbol{g}](x) = -\boldsymbol{n} \times \mathbf{E}_{inc}(x), \quad x \in \Sigma. \tag{13}$$

Note that in equation (13), the operator $\mathcal{K}$ is defined by the formula (12) with the hypersingular integral on the right-hand side.

We solve these equations and substitute the obtained currents into formulas (9) and (4) to calculate the values of the secondary electric and magnetic fields at an arbitrary point outside the surface $\Sigma$.

## 1.2. Numerical Method

For the numerical solution of equation (13) we employ a widely used numerical scheme of the Galerkin method with piecewise linear basis functions (RWG functions). This numerical scheme was first described in [6] and its modern version is described in [2, 13].

The surface $\Sigma$ is approximated by a conformal system of triangular cells. Let $\tilde{\Sigma} = \bigcup_{i=1,N} \sigma_i$ be the approximation of the original total surface $\Sigma$.

The approximation of the unknown currents $g$ is a linear combination of RWG basis functions [6].

$$\boldsymbol{g}(x) = \sum_{i=1,M} g_i \boldsymbol{v}_i(x) \tag{14}$$

Each basis function $\boldsymbol{v}_i(x)$, $i = 1,...,M$ is associated with an edge common to two cells $\sigma_i^1$ and $\sigma_i^2$ and characterizes the flux of the vector field through this edge. If an edge is common to exactly two mesh cells, then one basis function from the list of functions with indices $i = 1,...,M$ is associated with this edge. If an edge is a junction of $p > 2$ cells $\sigma_{j_1},...,\sigma_{j_p}$ (these cells are numbered in a clockwise order), then $p-1$ basis functions are associated with this edge. Each basis function corresponds to a pair of cells $(\sigma_{j_1}, \sigma_{j_2})$, $(\sigma_{j_{p-1}}, \sigma_{j_p})$. Thus an ordered pair of cells $\sigma_i^1$ and $\sigma_i^2$ is associated with each basis function $\boldsymbol{v}_i(x)$, $i = 1,...,M$ (see [14]). Let $s_i^1$ and $s_i^2$ be the areas of cells $\sigma_i^1$ and $\sigma_i^2$. We also denote the vertices that lie opposite to the common edge of cells $\sigma_i^1$ and $\sigma_i^2$ as $C_i^1$ and $C_i^2$ for each basis function $\boldsymbol{v}_i(x)$.

The Bubnov–Galerkin method is used for the numerical solution of the boundary integral equations $(\boldsymbol{v}_i, \mathcal{K}\boldsymbol{v}_j)_{\tilde{\Sigma}} = (\boldsymbol{v}_i, \mathcal{K}[\tilde{\Sigma}, \boldsymbol{v}_j])_{\tilde{\Sigma}}$. The scalar product of two vector functions is understood as the integral over the surface $\Sigma$: $(\boldsymbol{g}, \boldsymbol{f})_{\tilde{\Sigma}} = \int_{\tilde{\Sigma}} (\boldsymbol{g}, \boldsymbol{f}) d\sigma$.

The scalar product with the electric operator is defined by the following expression:

$$(\boldsymbol{v}_i, \mathcal{K}\boldsymbol{v}_j)_{\tilde{\Sigma}} = \int\limits_{\sigma_i^1 \cup \sigma_i^2} \int\limits_{\sigma_j^1 \cup \sigma_j^2} \left(k^2 \boldsymbol{v}_i(x) \cdot \boldsymbol{v}_j(y) - D_i D_j\right) F(x-y) d\sigma_y d\sigma_x =$$

$$= \sum_{p=1,2} \sum_{q=1,2} \frac{(-1)^{p+q}}{s_i^p s_j^q} \int\limits_{\sigma_i^p} \int\limits_{\sigma_j^q} \left(k^2 (C_i^p - x) \cdot (C_j^q - y) - 4\right) \frac{e^{\mathbf{i}kr}}{r} d\sigma_y d\sigma_x. \quad (15)$$

To compute the integral in expression (15), we use formulas with analytical extraction and integration of the singularity [2]. For regular integrals we developed an adaptive integration procedure based on Gaussian quadrature with subdivision of cells into smaller ones and accuracy control.

The system of boundary integral equations (13) is reduced to a system of linear algebraic equations.

$$\frac{\mathbf{i}}{k_0} \sum_{j=1}^{M} (\boldsymbol{v}_i, \mathcal{K}\boldsymbol{v}_j)_{\tilde{\Sigma}}\, g_j = -(\boldsymbol{v}_i, \mathbf{E}_{inc})_{\tilde{\Sigma}}, \quad i = 1, ..., M. \quad (16)$$

After the system (16) is solved for the variables $g_i$, $i = 1, ..., M$, the surface currents $\boldsymbol{g}$ are computed using formula (14).

### 1.3. Antenna Model

#### *1.3.1. Antenna excitation*

We use the well-known delta-gap model [2]. A port is a small gap of width $d$ between certain mesh edges. There is a potential difference in this gap. A port is called active if a voltage $U$ created by an EMF $\epsilon$ is applied across it. A port is called passive if a resistance $R$ is connected across it. It is possible to specify both voltage and resistance simultaneously, but such cases are not considered in this work. A port is approximated as a line consisting of one or several consecutive edges. Let $\Theta$ denote the set of indices $i$ for which the edge between cells $\sigma_i^1$ and $\sigma_i^2$ lies on the port line.

The primary field $\mathbf{E}_{inc}$ is defined as

$$\mathbf{E}_{inc} = \frac{U}{d} \boldsymbol{e}_0, \quad (17)$$

where $\boldsymbol{e}_0$ is the unit vector indicating the current direction. The voltage on each edge from the set of port edges $\Theta$ satisfies the formula

$$U = \epsilon - R(\boldsymbol{e}_0, \boldsymbol{g}_c), \quad (18)$$

where $\boldsymbol{g}_c$ is the value of the surface current on this edge.

In case of an active port the EMF $\epsilon$ is given and the resistance $R = 0$. The scalar product of the primary field and the basis function is the following:

$$(\boldsymbol{v}_i, \mathbf{E}_{inc})_{\tilde{\Sigma}} = \frac{U}{d} (\boldsymbol{v}_i, \boldsymbol{e}_0) = \frac{U}{d} \int\limits_{D} \frac{(C_i - x, \boldsymbol{e_0})}{s_i} d\sigma = \frac{U}{d} \int\limits_{D} \frac{2}{L} d\sigma = 2U, \quad i \in \Theta, \quad (19)$$

$D$ is the part of the surface $\Sigma$ where the gap is located. If the $i$-th edge does not belong to the set $\Theta$, then $(\boldsymbol{v}_i, E_{inc})_{\tilde{\Sigma}} = 0$. That is, the right-hand side is a vector consisting of two values: 0 or $2U$. In case of a passive port $(\boldsymbol{v}_i, E_{inc}) = 0$ for $i \notin \Theta$, as well for the active port.

Now let us consider the passive port edges

$$(\boldsymbol{v}_i, \mathbf{E}_{inc})_{\tilde{\Sigma}} = 2U = -2R(\boldsymbol{e}_0, \boldsymbol{g}_c) = \; = -2R \sum_{k\in\Theta} g_k(\boldsymbol{e}_0, \boldsymbol{v}_i) = -4R \sum_{k\in\Theta} g_k, \quad i \in \Theta. \tag{20}$$

This term is not included into the right-hand side. It is added to the system of equations as a sparse matrix $B$ where $b_{ik} = -4R$ if $i, k \in \Theta$.

### *1.3.2. Current and impedance*

Important antenna characteristics are the port current, input impedance, S-parameters, and standing wave ratio (VSWR). The input impedance of an antenna is the complex ratio of the voltage to the current at its feed point. S-parameters (scattering parameters) describe how electromagnetic power propagates through a multi-port network, quantifying reflection and transmission coefficients. The VSWR is a measure of the impedance mismatch between the antenna and its feed line, defined as the ratio of the maximum to the minimum voltage of the standing wave along the line.

First we consider a single antenna. The current through the port is the flux of the vector $g$ through the line consisting of all port edges

$$Y = (\boldsymbol{e}_0, \boldsymbol{g}_c) = 2 \sum_{k\in\Theta} g_k. \tag{21}$$

We denote $Z$ as the port impedance, $S$ as the S-parameter, $\eta$ as the VSWR, and $R_0$ as the matching impedance (a given value). According to [5], the following formulas are used for these characteristics:

$$Z = \frac{U}{Y}, \tag{22}$$

$$S = \frac{Z - R_0}{Z + R_0}, \tag{23}$$

$$\eta = \frac{1 + |S|}{1 - |S|}. \tag{24}$$

Now we consider an antenna system consisting of $m$ active ports. Each port consists of a set of edges $\Theta_m$. In this case, the calculation of the antenna system characteristics is performed according to the following scheme. Each port is sequentially considered active with an EMF $\epsilon = 1$ V, while the others are set as passive with the same resistance $R_0$. The currents $Y_{ij}$ are computed:

$$Y_{ij} = 2 \sum_{k\in\Theta_i} g_k, \quad \epsilon \neq 0 \text{ on } \Theta_j.$$

As a result, a current matrix $Y$ is obtained. The mutual impedance matrix Z, S-parameter matrix, and VSWR values are calculated using the formulas [5]:

$$Z = Y^{-1}, \tag{25}$$

$$S = \left(\frac{Z}{R_0} + I\right)^{-1} \left(\frac{Z}{R_0} - I\right), \tag{26}$$

$$\eta_i = \frac{1 + |S_{ii}|}{1 - |S_{ii}|}. \tag{27}$$

Here $Y$, $Z$ and $S$ are complex matrices of size $m \times m$, and $\eta$ is a real vector of size $m$.

### *1.3.3. Optimization of calculations*

Now let us consider an antenna system with a number of ports $m > 1$, for which it is required to compute the mutual impedance, S-parameter and VSWR of all ports. In the resulting system of equations, the main part of the matrix is independent of the ports and is assembled according to formula (16). However, when passive ports are present, the addition term (20) appears. If the problem is solved "head-on", the matrix is calculated $m$ times and the system is solved $m$ times. To avoid an increase in the computational complexity of the problem, we used a well-known formula from linear algebra, namely the Woodbury formula, which is a generalization of Sherman–Morrison formula [3].

Let us introduce the notation: $A \in \mathbb{C}^{n\times n}$, $U \in \mathbb{C}^{n\times m}$, $V \in \mathbb{C}^{k\times n}$, and $I_m \in \mathbb{R}^{m\times m}$ is the identity matrix.

$$B = A + UV \ \Rightarrow B^{-1} = A^{-1} - A^{-1}U(I_m + VA^{-1}U)^{-1}VA^{-1}. \tag{28}$$

If the rank of the correction is $m \ll n$, then this formula has an asymptotic complexity of $O(nm^2 + m^3)$, which is insignificant compared to complexity of matrix invertion $O(n^3)$. So we can get all the required antenna characteristics with $O(n^3)$ instead of $O(mn^3)$ in case of sequentially solving the problems.

However for large linear systems, constructing the inverse matrix $A^{-1}$ is impossible. The problem is solved approximately using one or another iterative method. To apply the Woodbury formula to the iterative method for solving the problem described above, we do the following. For each port we form an indicator vector $u_p = \{0,1\}^n$, $p \in \{1,\ldots,m\}$, where ones are only in those rows that correspond to the edges of this port. If this port is the only passive one, and its input impedance is $R_0$, then the system matrix is equal to

$$B = A - 4R_0 u_p u_p^T. \tag{29}$$

Since the ports do not intersect, $U = [u_1,\ldots,u_m]$ is a matrix with linearly independent columns. Let us split $U$ into $u_p$ and $\tilde{U}_p = [u_1,\ldots,u_{p-1},u_{p+1},\ldots,u_m]$. If port $p$ is the only active one, then the system takes the form

$$(A - 4R_0\, \tilde{U}_p \tilde{U}_p^T)y = -2u_p. \tag{30}$$

The construction of the solution for all ports is carried out according to Algorithm 1. This algorithm is applicable to both direct and iterative methods for solving the system $AY = U$. Thus, instead of solving $m$ systems with different matrices, it is sufficient to solve one system with $m$ right-hand sides and compute the currents using the specified formula.

**Algorithm 1** Computation of currents using the Woodbury formula

Solve $AY = U$
$X \leftarrow Y * 2R_0$
**for** $p = 1$ to $m$ **do**
    $x_p, \tilde{X}_p \leftarrow X$
    $g_p \leftarrow x_p - \tilde{X}_p \left(I_{m-1} + \tilde{U}_p^T \tilde{X}_p\right)^{-1} \tilde{U}_p^T x_p$
**end for**
**return** $g$

## 2. Results

The calculations were performed using the authors' program "edmpis" [4]. This program allows to solve the problems of electromagnetic wave scattering and antenna radiation not only for perfectly conducting but also for dielectric objects. For small-scale problems (up to 40,000 equations), a direct LU-decomposition method from LAPACK library is employed. For large-scale problems, the matrix is computed using the mosaic-skeleton approximation method [1, 8], and the system of linear algebraic equations is solved using the minimal residual method optimized for multiple right-hand sides [9].

The "edmpis" program uses MPI and OpenMP libraries to accelerate computations. When using the direct LU solver, matrix assembly is accelerated using OpenMP. When using the iterative solver with matrix approximation, MPI is used, and to a lesser extent, OpenMP. The testing of acceleration of parallel program was performed in [8].

It should be noted that the fast multipole algorithm is considered to be the classical method for compression of dense matrices in the electrodynamic problems. We use the mosaic-skeleton approximation because of its universality and good compression. The testing of acceleration of mosaic-skeleton approximation [12] shows almost linear speedup with the increase of MPI processes.

The calculations were performed on the cluster of the Marchuk Institute of Numerical Mathematics, Russian Academy of Sciences. The main cluster specifications are provided below.
Computational nodes "normal":

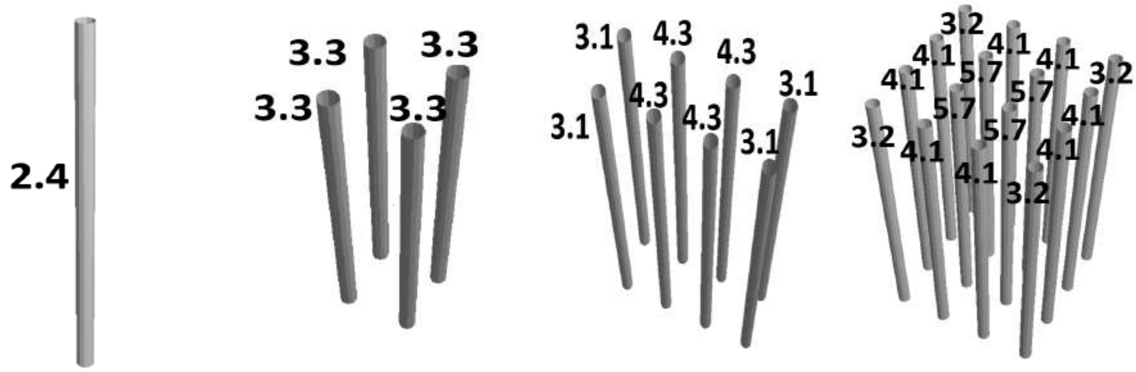
- 40 cores (two 20-core Intel Xeon Gold 6230@2.10GHz processors);
- 384 GB RAM;
- 480 GB SSD storage;
- Operating system: SUSE Linux Enterprise Server 15 SP6 (x86_64).

Computational nodes "short":

- 24 cores (two 12-core Intel Xeon Silver 4214@2.20GHz processors);
- 128 GB RAM;
- 480 GB SSD storage;
- Operating system: SUSE Linux Enterprise High Performance Computing 15 SP4 (x86_64).

### 2.1. Wire Antenna

As a simple example we consider a thin wire with a radius of 1 cm and a length of 50 cm. Such a wire can be considered as a linear antenna. The calculation was performed on a single "short" type node using 24 cores. In this case the system of linear equations was solved by a direct method.




(a) wire1x1 (b) wire2x2 (c) wire4x2 (d) wire4x4

**Figure 1.** Antenna arrays composed of wire antennas and their VSWR

Figure 1 shows a single antenna ("strip1x1") and three antenna systems obtained by copying this antenna: "strip2x2", "strip4x2", "strip4x4". The numbers indicate the calculated VSWR values of the antenna system elements in the case where this element is active and the other elements are passive.

**Table 1.** Program execution time for wire antennas

| Geometry | $m$ | $N_{eq}$ | $T_W$ | $T_0$ |
|---|---|---|---|---|
| wire1x1 | 1 | 1788 | 5.35 | 5.31 |
| wire2x2 | 4 | 7152 | 31.03 | 131.48 |
| wire4x2 | 8 | 14304 | 90.77 | 730.25 |
| wire4x4 | 16 | 28608 | 317.01 | 5169.51 |

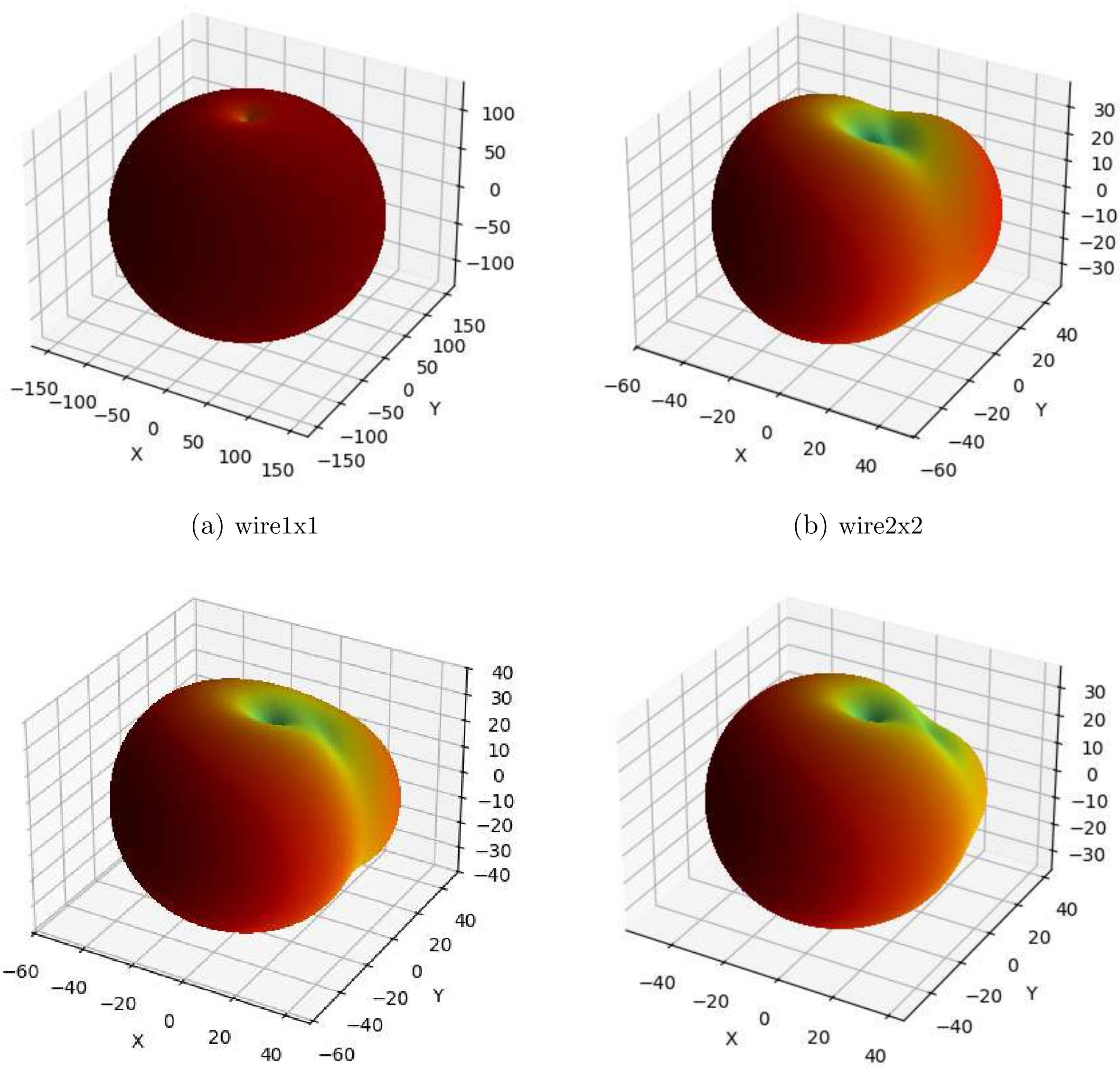
Table 1 shows the program execution time for calculations without optimization ($T_0$) and with optimization using the Woodbury formula ($T_W$). Time is measured in seconds. $N_{eq}$ is the number of equations. It can be noted that the optimization sped up the program by a factor of $m$ as expected.

For illustration, Fig. 2 shows the radiation patterns of the antenna systems at a frequency of 0.3 GHz for one of the port loading variants. In all cases, the only active antenna is the one in the lower left corner of Fig. 1. The radiation patterns reflect the dependence of the radiation power asymptote at large distances on the direction to the receiver:

$$F = \lim_{R\to\infty} \frac{4\pi|\boldsymbol{E}(x)|^2}{R^2}, \quad x = R\boldsymbol{\tau}, \tag{31}$$

where $\boldsymbol{\tau}$ is a unit vector indicating the direction to the receiver.

The radiation patterns, VSWR, and other characteristics calculated by the direct method and using the Woodbury formula coincide with a relative accuracy of $10^{-5}$. This is due only to rounding errors, since the Woodbury formula allows for an exact, not an approximate, solution.




(a) wire1x1 (b) wire2x2

(c) wire4x2 (d) wire4x4

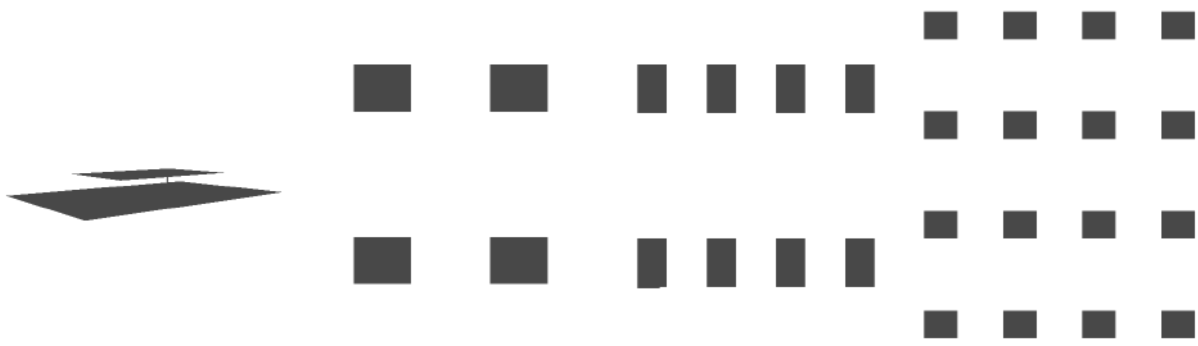
**Figure 2.** Radiation patterns of antenna arrays composed of wire antennas at a frequency of 0.3 GHz

## 2.2. Patch Antenna

Another example is a system of patch antennas. Each patch antenna consists of two plates. The excitation of the antenna is defined by a narrow strip connecting the antenna patches, where a lumped port is placed.

The calculation was performed on a single "normal" type node using 40 cores. The problem was solved by an iterative method with an accuracy of $10^{-2}$, which influenced the difference between the solution with optimization and without it.

Figure 3 shows a single patch antenna and three antenna arrays obtained by copying this antenna: "patch2x2", "patch4x2", "patch4x4". Unlike the previous example, the VSWR of the patch antennas is mostly independent of the array configuration and the position of the specific antenna. The VSWR of the active antenna is approximately 1.7 for all patch antennas in these



(a) patch1x1 (b) patch2x2 (c) patch4x2 (d) patch4x4

**Figure 3.** Antenna arrays composed of patch antennas

systems. The mutual influence of patch antennas on each other is significantly less than that of strip antennas. For the mutual impedances the relation $\frac{Z_{ij}}{Z_{ii}} \simeq 10^{-2}$, $i \neq j$ is typical.

Figure 4 shows the radiation patterns of the antenna arrays at a frequency of 4.25 GHz for one of the loading variants. In all cases the only active antenna is the one in the lower left corner. It can be observed how the increase in the number of antennas in the system increases the noise level of the radiation pattern.

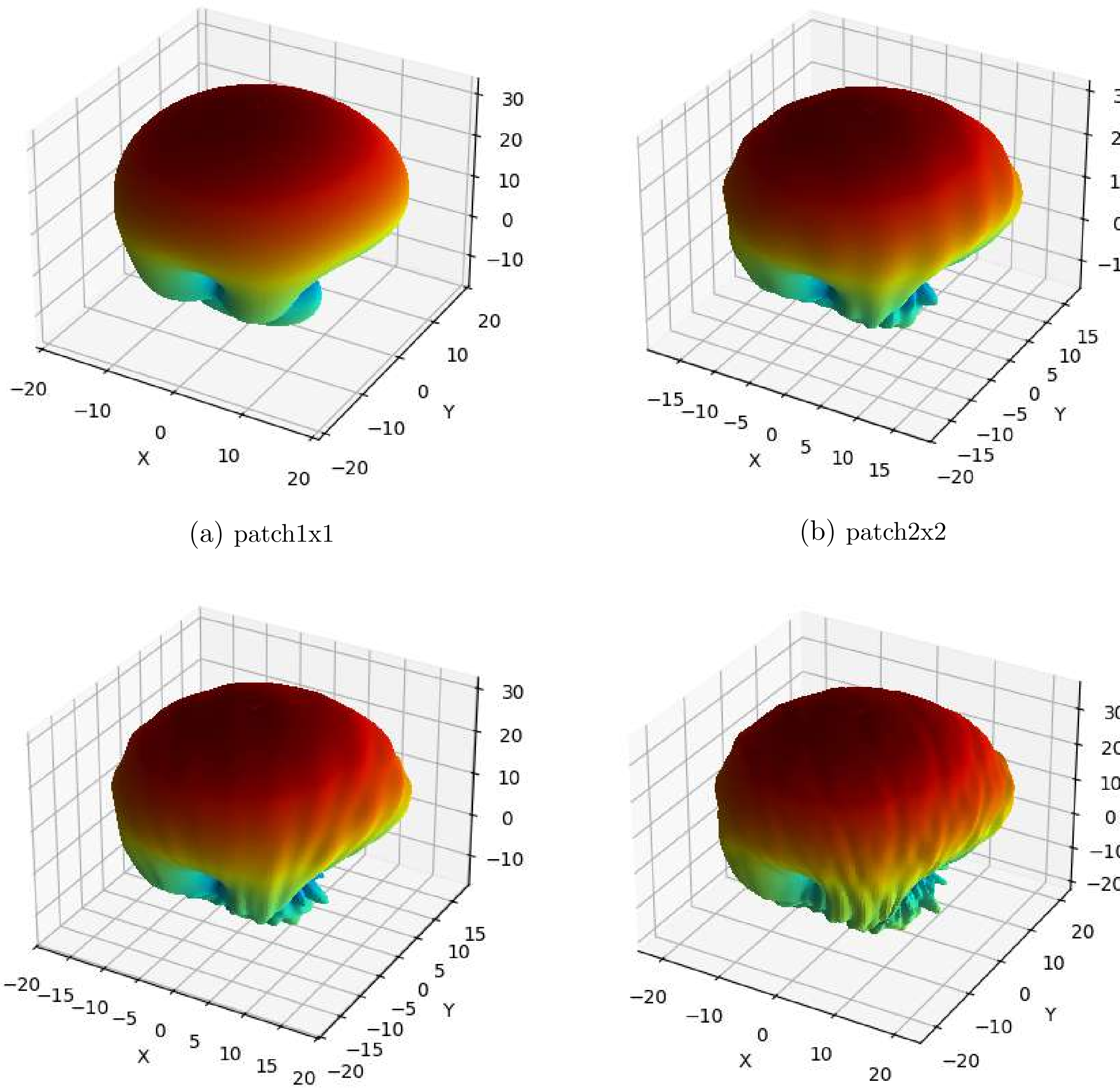
**Table 2.** Program execution time for patch antennas

| Geometry | $m$ | $N_{eq}$ | $T_W$ | $T_0$ | $I_W$ | $I_0$ |
|---|---|---|---|---|---|---|
| patch1x1 | 1 | 8879 | 38.18 | 38.16 | 323 | 323 |
| patch2x2 | 4 | 35516 | 181.32 | 610.9 | 1589 | 2592 |
| patch4x2 | 8 | 71032 | 506.92 | 2530.08 | 3285 | 6306 |
| patch4x4 | 16 | 142064 | 1829.72 | 9987.38 | 7071 | 14380 |

Table 2 shows the program execution time for calculations without optimization and with optimization using the Woodbury formula. $I_W$ is the number of GMRES iterations in the optimized program, $I_0$ is the total number of GMRES iterations when solving $m$ systems in the program without optimization. The radiation patterns, VSWR, and other characteristics coincide with the accuracy of $10^{-2}$, which corresponds to the accuracy of the iterative method.

## Conclusion

The purpose of this paper was to accelerate the calculation of characteristics of mutual coupling of elements in antenna arrays by the optimization with the Woodbury formula. The numerical experiment demonstrated that in case of direct method the execution time of the optimized program differs from the execution time of the program without optimization by approximately a factor of $m$, where $m$ is the number of antennas in the system (calculation of the linear antenna system). In the case of iterative method with a low-rank approximation of the system matrix (example with the patch antenna system), optimization also achieves a significant acceleration of computations, but with a smaller factor. For instance, the computation time for a system of 16 antennas decreased by approximately 5.5 times.




(a) patch1x1 (b) patch2x2

(c) patch4x2 (d) patch4x4

**Figure 4.** Radiation patterns of antenna arrays composed of patch antennas at a frequency of 4.25 GHz

The difference in the acceleration of computations when applying direct and iterative methods is as follows. In the direct algorithm, an $LU$ decomposition of the matrix is constructed using the LAPACK library, after which the problem is solved for one or $m$ right-hand sides in approximately the same time. For the patch antenna first the matrix is approximated and then the GMRES solver is called. When using optimization, the matrix approximation is performed once instead of $m$ times. Also, the used iterative solver is optimized for problems with multiple right-hand sides: the Krylov basis built for previous right-hand sides is used for the next one. However in this example the total number of iterations increases with the number of ports compared to the case of a single port, but with a smaller factor than the number of ports. When using optimization, the number of iterations decreases by 1.5 to 2 times compared to the total number of iterations without optimization. It is known that the right-hand side vectors $u_p$,

$p = 1, \ldots, m$, are orthogonal to each other, but this information is insufficient to predict the behavior of the iterative method on an arbitrary antenna array.

To sum up, the developed method sufficiently accelerates the calculation of characteristics of an antenna system with many active excitation elements.

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